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Two-dimensional real algebras with zero divisors

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BENJAMIN PEIRCE in his seminal work on linear associative algebras [6] classified all two-dimensional real associative (pure) algebras. He reported on his results from this work in talks delivered to the National Academy of Sciences during the period from 1867 to 1870, but they were not published until 1881, posthumously. (See [3], [4].) In 1958 LUCHIAN [4] began a classification of all two-dimensional real algebras with zero divisors, and in 1970 WALLACE [8] classified all two-dimensional power associative real algebras. Finally, in 1983 ALTHOEN and KUGLER [1] gave canonical forms for all two-dimensional real division algebras.

This paper complements all these previous works by presenting a list of canonical forms for multiplication tables of all two-dimensional real algebras which are not division algebras (i.e., which have zero divisors). The presentation is algorithmic: given any such algebra, one can easily derive one of our forms. Except in the case of the final table, as is clarified below, the tables presented are uniquely determined; the proof of this fact is not difficult and is omitted for purposes of brevity.

Consider the real algebra $\mathscr{A}$ with basis $\{\eta_1, \eta_2\}$ with respect to which multiplication is given by the following table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & a_1\eta_1+a_2\eta_2 & b_1\eta_1+b_2\eta_2 \\ \eta_2 & c_1\eta_1+c_2\eta_2 & d_1\eta_1+d_2\eta_2 \end{array} .$$

We use the convention that Roman letters represent real numbers and Greek letters represent elements of $\mathscr{A}$. We can write the table in the abbreviated form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

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by setting $\alpha=a_1\eta_1+a_2\eta_2$, etc. Let L_χ and R_χ denote, respectively, left and right translations by the element $\chi\in\mathscr{A}$:

$$L_\chi(\alpha) = \chi\alpha \quad \text{and} \quad R_\chi(\alpha) = \alpha\chi, \quad \text{for all} \quad \alpha\in\mathscr{A}.$$

Then for $\chi=x_1\eta_1+x_2\eta_2$, with $x_1, x_2\in\mathbf{R}$, we find that

$$\det(L_\chi) = Q_L(x_1, x_2) = M_L x_1^2+N_L x_1 x_2+P_L x_2^2 \quad \text{and}$$
$$\det(R_\chi) = Q_R(x_1, x_2) = M_R x_1^2+N_R x_1 x_2+P_R x_2^2,$$

where

$$M_L = |\alpha, \beta|, \quad N_L = |\alpha, \delta|+|\gamma, \beta|, \quad P_L = |\gamma, \delta|,$$
$$M_R = |\alpha, \gamma|, \quad N_R = |\alpha, \delta|+|\beta, \gamma|, \quad \text{and} \quad P_R = |\beta, \delta|,$$

given, for example, that the notation $|\alpha, \beta|$ denotes the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Let $\Delta=N_L^2-4M_LP_L=N_R^2-4M_RP_R$. By definition, $\mathscr{A}$ is a division algebra if and only if L_χ is nonsingular for all nonzero elements $\chi\in\mathscr{A}$ (or equivalently, R_χ is nonsingular for all nonzero elements $\chi\in\mathscr{A}$); this is the case if and only if the quadratic form $Q_L(x_1, x_2)$ is positive definite (or equivalently, $Q_R(x_1, x_2)$ is positive definite). Thus $\mathscr{A}$ is a division algebra if and only if $\Delta<0$, and $\mathscr{A}$ is an algebra with zero divisors if and only if $\Delta\geqq 0$.

As we proceed, we will subdivide the two-dimensional algebras with zero divisors into subclasses. One of the sets of criteria we use is the classification scheme of LUCHIAN [4]:

Definition. A two-dimensional algebra $\mathscr{A}$ belongs to the class:

L_1, if $\Delta > 0$,	R_1, if $\Delta > 0$,
L_2, if $\Delta = 0$ but $Q_L(x_1, x_2) \not\equiv 0$,	R_2, if $\Delta = 0$ but $Q_R(x_1, x_2) \not\equiv 0$,
L_3, if $Q_L(x_1, x_2) \equiv 0$,	R_3, if $Q_R(x_1, x_2) \equiv 0$.

Remarks. (1) The quadratic forms $Q_L(x_1, x_2)$ and $Q_R(x_1, x_2)$ and the quantity Δ are dependent upon the basis $\{\eta_1, \eta_2\}$ of $\mathscr{A}$, and at first glance it appears that the subclasses L_i and R_j are as well. However, this is not the case. In fact, the algebra $\mathscr{A}$ is in the class:

a) $L_1(R_1)$ if and only if there are two independent left (right) zero divisors in $\mathscr{A}$, but not every nonzero element of $\mathscr{A}$ is a left (right) zero divisor;

b) $L_2(R_2)$ if and only if there exists a left (right) zero divisor $\chi\in\mathscr{A}$ and every other left (right) zero divisor in $\mathscr{A}$ is a multiple of χ;

c) $L_3(R_3)$ if and only if every nonzero element of $\mathscr{A}$ is a left (right) zero divisor.

(2) From the theory of quadratic forms (see [5], pp. 85—86), we know that if $\{\nu_1, \nu_2\}$ is another basis for $\mathscr{A}$, generating corresponding quadratic forms $Q_L^0(x_1, x_2)$ and $Q_R^0(x_1, x_2)$ and quantity Δ^0, and if $T\colon \mathscr{A}\to\mathscr{A}$ is a linear transformation such

that $T(\eta_i)=\nu_i$, then $\Delta^0=|T|^2\Delta$. This provides a direct proof that the class $L_1=R_1$ is independent of basis.

(3) The triplets L_1, L_2, L_3 and R_1, R_2, R_3 each form partitions of the class of algebras with zero divisors. Following the notation of Luchian, we let L_iL_j denote the intersection $L_i \cap L_j$. It is clear that $L_1=R_1$, and hence $L_1R_2=L_1R_3=L_2R_1=$ $=L_3R_1=\emptyset$; all other intersections, however, are nonempty.

As seen above, if $\Delta \geqq 0$, there exist nontrivial elements $\chi, \psi \in \mathcal{A}$ such that $L_\chi(\psi)=\chi\psi=0$. In fact, in some cases there exists a nontrivial element $\chi \in \mathcal{A}$ such that $L_\chi(\chi)=\chi^2=0$. The second set of criteria we use for classification of the two-dimensional algebras with zero divisors is based upon the number of such elements (if any exist).

Definition. A two-dimensional algebra $\mathcal{A}$ belongs to the class:

S, if there exists a nontrivial element $\chi \in \mathcal{A}$ whose square is 0,

N, if no such element exists.

The following proposition gives a criterion which determines whether an algebra lies in the class N or the class S.

Proposition. *A two-dimensional real algebra $\mathcal{A}$ lies in the class N if and only if given any multiplication table for $\mathcal{A}$:*

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

we have

$$\begin{vmatrix} a_1 & b_1+c_1 & d_1 & 0 \\ 0 & a_1 & b_1+c_1 & d_1 \\ a_2 & b_2+c_2 & d_2 & 0 \\ 0 & a_2 & b_2+c_2 & d_2 \end{vmatrix} \neq 0.$$

Proof. There exists a nontrivial element $\chi=x_1\eta_1+x_2\eta_2$ in $\mathcal{A}$ with $\chi^2=0$ if and only if there exists a nontrivial solution (x_1, x_2) to the system:

$$a_1x_1^2+(b_1+c_1)x_1x_2+d_1x_2^2=0,$$

$$a_2x_1^2+(b_2+c_2)x_1x_2+d_2x_2^2=0.$$

This will be the case if and only if the two polynomials

$$a_1z^2+(b_1+c_1)z+d_1 \quad \text{and} \quad a_2z^2+(b_2+c_2)z+d_2$$

have a common root, and this is true if and only if their resultant, given by the determinant above, is 0. (See [7] for details.)

Suppose that $\mathscr{A}$ is an algebra in which there exists a nontrivial element $\chi\in\mathscr{A}$ such that $L_\chi(\chi)=\chi^2=0$. In this case, we can set $\eta_1=\chi$ and let η_2 be any element independent of χ to obtain a multiplication table in the form of Table S below.

Otherwise, $\chi\psi=0$ implies that χ and ψ are independent elements of $\mathscr{A}$, and setting $\eta_1=\chi$ and $\eta_2=\psi$, we arrive at a multiplication table in the form of Table N below. Note that since $\mathscr{A}$ has no nontrivial elements which square to 0, necessarily $x_1^2\alpha+x_1x_2\gamma+x_2^2\gamma^2=0$ if and only if $x_1=x_2=0$. (This implies, in particular, that $\alpha, \delta\neq 0$.)

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & \gamma & \delta \end{array} \qquad \begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & 0 \\ \eta_2 & \gamma & \delta \end{array} \qquad (x_1^2\alpha+x_1x_2\gamma+x_2^2\delta=0 \Leftrightarrow x_1=x_2=0).$$

Table S *Table N*

The classes N and S clearly also form a partition of the class of two-dimensional algebras with zero divisors. The multiplication tables given above are particularly useful in that they simplify the definitions of the L_j and R_j classes given above, as is shown by the following proposition. The proof follows immediately from the definitions of the quadratic forms Q_L and Q_R and the quantity Δ.

Proposition. *If an algebra $\mathscr{A}$ has a basis with respect to which its multiplication table has the form of Table S, then $\mathscr{A}$ belongs to:*

L_1, *if* $|\beta, \gamma|\neq 0$, R_1, *if* $|\beta, \gamma|\neq 0$,

L_2, *if* $|\beta, \gamma|=0$ *but* $|\gamma, \delta|\neq 0$, R_2, *if* $|\beta, \gamma|=0$ *but* $|\beta, \delta|\neq 0$,

L_3, *if* $|\beta, \gamma|=|\gamma, \delta|=0$, R_3, *if* $|\beta, \gamma|=|\beta, \delta|=0$.

If an algebra $\mathscr{A}$ has a basis with respect to which its multiplication table has the form of Table N, then $\mathscr{A}$ belongs to:

L_1, *if* $|\alpha, \delta|\neq 0$, R_1, *if* $|\alpha, \delta|\neq 0$,

L_2, *if* $|\alpha, \delta|=0$ *but* $|\gamma, \delta|\neq 0$, R_2, *if* $|\alpha, \delta|=0$ *but* $|\alpha, \gamma|\neq 0$,

L_3, *if* $|\alpha, \delta|=|\gamma, \delta|=0$, R_3, *if* $|\alpha, \gamma|=|\alpha, \delta|=0$.

We now find a set of canonical forms for multiplication tables of two-dimensional algebras with zero divisors. As we proceed, we further partition the classes L_iR_j defined by Luchian via their intersections with the classes N and S. Following the previously established notation, we let concatenation denote intersection.

The Case L_3R_3: It is clear from the original definitions of the classes L and R that $\mathscr{A}$ belongs to the class L_3R_3 if and only if in any multiplication table for $\mathscr{A}$, the elements α, β, γ and δ are pairwise dependent. This implies that any multiplication for $\mathscr{A}$ has the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha v & bv \\ \eta_2 & cv & dv \end{array}.$$

The Case L_3R_3S: In this case we can begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & bv \\ \eta_2 & cv & dv \end{array}.$$

The zero algebra is clearly one element of this class:

$$(L_3R_3S/1) \quad \begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & 0 \\ \eta_2 & 0 & 0 \end{array}.$$

This algebra is not catalogued by Peirce since it is "mixed". (See [6], p. 100.) It is Wallace's power associative algebra A_1.

Assume in what follows that the multiplication on $\mathscr{A}$ is nontrivial, so that $v \neq 0$ and at least one of b, c and d is nonzero. The element v is determined up to scalar multiples by the fact that it spans the ranges of the left and right multiplication maps L_χ and R_χ of any element $\chi \in \mathscr{A}$. We proceed by considering two cases: $v^2=0$ and $v^2 \neq 0$.

(1) $v^2=0$: Take $\eta_1 = v$ and η_2 any element independent of v to get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & c\eta_1 & d\eta_1 \end{array}.$$

(1a) $b \neq 0$, $b+c \neq 0$: Set $\zeta_1 = \eta_1$, $\zeta_2 = (-d/b[b+c])\eta_1 + (1/b)\eta_2$. Then with $c' = c/b$, we get the table:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & 0 \end{array} \quad (c' \neq -1).$$

(1b) $b \neq 0$, $b+c=d=0$: Set $\zeta_1 = \eta_1$, $\zeta_2 = (1/b)\eta_2$ to get:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & 0 \end{array}.$$

In the two preceding cases we have shown that we can get a table in the form:

$$(L_3R_3S/2)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & 0 \end{array}\ .$$

When $c'=-1$, this is Wallace's algebra $A(-1)$.

(1c) $b\neq 0,\ b+c=0,\ d\neq 0$: Set $\zeta_1=(d/b^2)\eta_1,\ \zeta_2=(1/b)\eta_2$ to get:

$$(L_3R_3S/3)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & \zeta_1 \end{array}\ .$$

(1d) $b=c=0\ (d\neq 0)$: Set $\zeta_1=d\eta_1,\ \zeta_2=\eta_2$ to get:

$$(L_3R_3S/4)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & 0 & \zeta_1 \end{array}\ .$$

This is Peirce's associative algebra (c_2) and Wallace's algebra B_1.

(1e) $b=0,\ c\neq 0$: Set $\zeta_1=\eta_1,\ \zeta_2=(-d/c^2)\eta_1+(1/c)\eta_2$ to get:

$$(L_3R_3S/5)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_1 & 0 \end{array}\ .$$

(2) $v^2\neq 0$: Take $\eta_2=v$ and η_1 any element such that $\eta_1^2=0$ (so that η_1 is necessarily independent of v). We get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & c\eta_2 & d\eta_2 \end{array}\qquad (d\neq 0).$$

(2a) $b\neq 0,\ |c/b|\leqq 1$: Set $\zeta_1=(1/b)\eta_1,\ \zeta_2=(1/d)\eta_2$. Then with $c'=c/b$, we get:

$$(L_3R_3S/6)\qquad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & c'\zeta_2 & \zeta_2 \end{array}\qquad (|c'|\leqq 1).$$

(2b) $b\neq 0,\ |c/b|\geqq 1$: Set $\zeta_1=(-1/c)\eta_1+([b+c]/cd)\eta_2$ and $\zeta_2=(1/d)\eta_2$. With $c'=b/c$, we again get a table in the form of $(L_3R_3S/6)$.

(2c) $b=0,\ c\neq 0$: Set $\zeta_1=(-1/c)\eta_1+(1/d)\eta_2,\ \zeta_2=(1/d)\eta_2$, and $c'=0$ to get yet another table in the form of $(L_3R_3S/6)$.

(2d) $b=c=0$: Set $\zeta_1=\eta_1$, $\zeta_2=(1/d)\eta_2$ to get:

$$(L_3 R_3 S/7) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & 0 & \zeta_2 \end{array} .$$

Although this algebra is associative, it is not catalogued by PEIRCE because it is "mixed". ([6], p. 100.) It is Wallace's algebra B_2.

The Case L_3R_3N: We begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & av & 0 \\ \eta_2 & cv & dv \end{array}$$

where $v\neq 0$ and $(0, 0)$ is the unique solution of $ax_1^2+cx_1x_2+dx_2^2=0$. Once again, the element v is determined up to scalar multiples by the fact that it spans the ranges of the left and right multiplication maps L_χ and R_χ of any element $\chi\in\mathscr{A}$. If $\{\eta_1, v\}$ is dependent, we replace η_1 by v to arrive at the table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & a\eta_1 & 0 \\ \eta_2 & c\eta_1 & d\eta_1 \end{array} .$$

Otherwise, $v=n_1\eta_1+n_2\eta_2$, where $n_2\neq 0$. In this case, if $\chi=(-dn_2)\eta_1+(an_1+cn_2)\eta_2$, then $v\chi=0$ and $|v, \chi|=\begin{vmatrix} n_1 & n_2 \\ -dn_2 & an_1+cn_2 \end{vmatrix}=an_1^2+cn_1n_2+dn_2^2\neq 0$, so that $\{v, \chi\}$ is independent. Thus by replacing η_1 by v and η_2 by χ, we can again assume that we have a table in the preceding form.

To proceed we need a definition:

Definition.

$$\operatorname{sgn} x = \begin{cases} 1 & \text{for} \quad x \geqq 0 \\ -1 & \text{for} \quad x < 0. \end{cases}$$

(Note that we do not define $\operatorname{sgn} 0=0$, as is customary.)

The fact that $(0, 0)$ is the unique solution of $ax_1^2+cx_1x_2+dx_2^2=0$ implies that $c^2-4ad<0$ and hence $ad>0$. Thus setting $\zeta_1=(1/a)\eta_1$, $\zeta_2=([\operatorname{sgn} c]/\sqrt{ad})\eta_2$, and $c'=|c|/\sqrt{ad}$, we arrive at the table:

$$(L_3 R_3 N) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1 & 0 \\ \zeta_2 & c'\zeta_1 & \zeta_1 \end{array} \quad (0 \leqq c' < 2).$$

Thus there are eight distinct table forms in the L_3R_3 case.

The Case L_3R_2S: In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

with the L_3 and R_2 conditions

$$|\beta, \gamma| = |\gamma, \delta| = 0, \quad |\beta, \delta| \neq 0.$$

These imply that $\gamma=0$ and the table in fact has the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & 0 & \delta \end{array} \quad (|\beta, \delta| \neq 0).$$

There are two cases to consider: that $\{\eta_1, \beta\}$ is dependent, and that $\{\eta_1, \beta\}$ is independent.

(1) $\{\eta_1, \beta\}$ is dependent: Our table becomes:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & 0 & d_1\eta_1+d_2\eta_2 \end{array} \quad (bd_2 \neq 0).$$

(1a) $b-d_2\neq 0$: Take $\zeta_1=\eta_1$ and $\zeta_2=[d_1/b(d_2-b)]\eta_1+(1/b)\eta_2$. Then with $d'= =d_2/b$, we arrive at the table:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & d'\zeta_2 \end{array} \quad (d' \neq 0, 1).$$

(1b) $b-d_2=0=d_1$: Set $\zeta_1=\eta_1$ and $\zeta_2=(1/b)\eta_2$ to arrive at:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & \zeta_2 \end{array}.$$

Both of the preceding tables are of the form:

$$(L_3R_2S/1) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & d'\zeta_2 \end{array} \quad (d' \neq 0).$$

If $d'=1$, this algebra is associative, but it was missed by Peirce. It is Wallace's algebra A_3.

(1c) $b-d_2=0,\ d_1\neq 0$: Set $\zeta_1=(d_1/b^2)\eta_1$ and $\zeta_2=(1/b)\eta_2$ to arrive at:

$$(L_3R_2S/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & 0 & \zeta_1+\zeta_2 \end{array}\quad .$$

(2) $\{\eta_1, \beta\}$ is independent: Replace η_2 by β to get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & 0 & d_1\eta_1+d_2\eta_2 \end{array}\quad (bd_1\neq 0).$$

Taking $\zeta_1=(1/b)\eta_1$, $\zeta_2=[\operatorname{sgn}(d_2)/\sqrt{|bd_1|}]\eta_2$, and $d'=|d_2|/\sqrt{|bd_1|}$, we arrive at the table:

$$(L_3R_2S/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & 0 & \pm\zeta_1+d'\zeta_2 \end{array}\quad (d'\geqq 0).$$

The Case L_3R_2N: In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & 0 \\ \eta_2 & \gamma & \delta \end{array}\quad (\alpha, \delta\neq 0)$$

where the L_3 and R_2 conditions on such a table imply that

$$|\alpha, \delta| = |\gamma, \delta| = 0 \quad\text{and}\quad |\alpha, \gamma|\neq 0.$$

The only way these can hold is if $\delta=0$, but this is not allowed in Case N. Thus $L_3R_2N=\emptyset$.

The Case L_2R_3: Recall that every algebra $\mathscr{A}$ has a corresponding algebra $\mathscr{A}^{\text{opp}}$ whose underlying set and addition is the same as for $\mathscr{A}$ and whose multiplication $\circ$ is defined by $\alpha\circ\beta=\beta\alpha$. It follows that $\{\eta_1, \eta_2\}$ is a basis for $\mathscr{A}$ giving the multiplication table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

if and only if $\{\eta_1, \eta_2\}$ is also a basis for $\mathscr{A}^{\text{opp}}$ giving the multiplication table:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & \gamma \\ \eta_2 & \beta & \delta \end{array}\quad .$$

Moreover, $\mathscr{A}$ lies in the class $L_i(R_i)$ if and only if $\mathscr{A}^{\text{opp}}$ lies in the class $R_i(L_i)$, for

$i=1, 2, 3$. Thus if $\mathscr{A}$ lies in the class L_2R_3, $\mathscr{A}^{\text{opp}}$ must lie in the class L_3R_2. In particular, it follows that $L_2R_3N=\emptyset$.

The Case L_2R_3S: If $\mathscr{A}$ lies in the class L_2R_3S, then $\mathscr{A}^{\text{opp}}$ lies in the class L_3R_2S and hence must have a basis giving a multiplication table in one of the canonical forms just presented. Thus with respect to the same basis, $\mathscr{A}$ has a table in one of the following forms:

$$(L_2R_3S/1)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_1 & d'\zeta_2 \end{array} \quad (d' \neq 0).$$

If $d'=1$, this is algebra is associative. It is Peirce's algebra (b_2) and Wallace's algebra $A_2=A(0)$.

$$(L_2R_3S/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_1 & \zeta_1+\zeta_2 \end{array}$$

$$(L_2R_3S/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & 0 \\ \zeta_2 & \zeta_2 & \pm\zeta_1+d'\zeta_2 \end{array} \quad (d' \geqq 0).$$

We have shown that there are three distinct table forms in each of the L_3R_2 and L_2R_3 cases.

The Case L_2R_2S: In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

where the L_2 and R_2 conditions imply that

$$|\beta, \gamma| = 0 \quad \text{and} \quad |\beta, \delta|, |\gamma, \delta| \neq 0.$$

This implies that the table must actually have the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & c\beta & \delta \end{array} \quad (c \neq 0,\ |\beta, \delta| \neq 0).$$

Note that in this case $(x_1\eta_1+x_2\eta_2)^2=x_2[(1+c)x_1\beta+x_2\delta]$. Since $\{\beta, \delta\}$ is independent, this implies that $(x_1\eta_1+x_2\eta_2)^2=0$ if and only if $x_2=0$. This means that η_1 is determined up to scalars, as is β, since β spans the range of L_{η_1}. Again there are two cases to consider: when $\{\eta_1, \beta\}$ is dependent, and when $\{\eta_1, \beta\}$ is independent.

(1) $\{\eta_1, \beta\}$ is dependent: Our table becomes:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & c\eta_1 & d_1\eta_1+d_2\eta_2 \end{array} \quad (bcd_2 \neq 1).$$

(1a) $b+c-d_2 \neq 0$: Take $\zeta_1=\eta_1$ and $\zeta_2=[d_1/b(d_2-b-c)]\eta_1+(1/b)\eta_2$. With $c'=c/b$ and $d'=d_2/b$, we arrive at the table:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & d'\zeta_2 \end{array} \quad (c'd' \neq 0,\ d'-c' \neq 0).$$

(1b) $b+c-d_2=0=d_1$: Set $\zeta_1=\eta_1$ and $\zeta_2=(1/b)\eta_2$. With $c'=c/b$ and $d'=$ $=d_2/b$, we arrive at:

$$\begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & d'\zeta_2 \end{array} \quad (c'd' \neq 0,\ d' = c'+1).$$

Both of the preceding tables are of the form:

$$(L_2R_2S/1) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & d'\zeta_2 \end{array} \quad (c'd' \neq 0).$$

When $c'=d'=1$, we obtain Peirce's associative algebra (a_2) and Wallace's algebra B_5. When $c'=1/\sigma$ and $d'=(1+\sigma)/\sigma$ ($\sigma \neq 0, -1$), we obtain Wallace's algebra $A(\sigma)$.

(1c) $b+c-d_2=0$, $d_1 \neq 0$: Set $\zeta_1=(d_1/b^2)\eta_1$ and $\zeta_2=(1/b)\eta_2$. Then with $c'=$ $=c/b$, we get:

$$(L_2R_2S/2) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 & \zeta_1+(1+c')\zeta_2 \end{array} \quad (c'(1+c') \neq 0).$$

2) $\{\eta_1, \beta\}$ is independent: Here we can replace η_2 by β to get a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & c\eta_2 & d_1\eta_1+d_2\eta_2 \end{array} \quad (bcd_1 \neq 0).$$

In this case, take $\zeta_1=(1/b)\eta_1$ and $\zeta_2=[\operatorname{sgn}(d_2)/\sqrt{|bd_1|}]\eta_2$. With $c'=c/b$ and $d'=|d_2|/\sqrt{|bd_1|}$ we arrive at the table:

$$(L_2R_2S/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & c'\zeta_2 & \pm\zeta_1+d'\zeta_2 \end{array} \quad (c'\neq 0,\ d'\geqq 0).$$

The Case L_2R_2N: Here we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & 0 \\ \eta_2 & \gamma & \delta \end{array} \quad (\alpha, \delta \neq 0)$$

where the L_2 and R_2 definitions imply that

$$|\alpha, \delta| = 0 \quad \text{and} \quad |\alpha, \gamma|, |\gamma, \delta| \neq 0.$$

Since $\alpha\neq 0$ and $\{\alpha, \delta\}$ is dependent, we may rewrite the table as:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & \alpha & 0 \\ \eta_2 & \gamma & d\alpha \end{array} \quad (d\neq 0,\ |\alpha, \gamma| \neq 0).$$

For $\chi=x_1\eta_1+x_2\eta_2$ and $\xi=y_1\eta_1+y_2\eta_2$, $\chi\xi=(x_1y_1+dx_2y_2)\alpha+x_2y_1\gamma$. Thus since $\{\alpha, \gamma\}$ is independent, $\chi\xi=0$ if and only if $x_1y_1+dx_2y_2=x_2y_1=0$. When $\chi, \xi\neq 0$, this is the case if and only if $x_2=y_1=0$. Hence the only left zero divisors in $\mathscr{A}$ are multiples of η_1 and the only right zero divisors in $\mathscr{A}$ are multiples of η_2. (It follows that η_1 and η_2 are determined up to scalar multiples.)

We now rewrite the table as:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & a_1\eta_1+a_2\eta_2 & 0 \\ \eta_2 & c_1\eta_1+c_2\eta_2 & d(a_1\eta_1+a_2\eta_2) \end{array} \quad \left(d\neq 0,\ \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \neq 0\right).$$

(1) $a_1, a_2\neq 0$: Take $\zeta_1=(1/a_1)\eta_1$, $\zeta_2=(a_2/a_1^2)\eta_2$. With $c_1'=c_1a_2/a_1^2$, $c_2'=c_2/a_1$, and $d'=da_2^2/a_1^2$, we get the table:

$$(L_2R_2N/1)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1+\zeta_2 & 0 \\ \zeta_2 & c_1\zeta_1+c_2'\zeta_2 & d'(\zeta_1+\zeta_2) \end{array} \quad (d'\neq 0,\ c_1\neq c_2').$$

(2) $a_2=0\,(a_1\neq 0)$: Let $\varepsilon=\operatorname{sgn}[c_1/a_1]/(a_1\sqrt{|d|})$ and take $\zeta_1=(1/a_1)\eta_1$ and $\zeta_2=\varepsilon\eta_2$. With $c_1'=|c_1/a_1|/\sqrt{|d|}$ and $c_2'=c_2/a_1$, we get:

$$(L_2R_2N/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1 & 0 \\ \zeta_2 & c_1'\zeta_1+c_2'\zeta_2 & \pm\zeta_1 \end{array}\quad (c_1'\geqq 0,\ c_2'\neq 0).$$

(3) $a_1=0\ (a_2\neq 0)$: Let $\varepsilon=\operatorname{sgn}[c_2/a_2]/(a_2\sqrt{|d|})$, $\zeta_1=\varepsilon\eta_1$ and $\zeta_2=(1/a_2|d|)\eta_2$, $c_1'=c_1/a_2|d|$ and $c_2'=|c_2/a_2|/\sqrt{|d|}$ to get the table:

$$(L_2R_2N/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_2 & 0 \\ \zeta_2 & c_1'\zeta_1+c_2'\zeta_2 & \pm\zeta_2 \end{array}\quad (c_1'\neq 0,\ c_2'>0).$$

Thus there are six table forms in the L_2R_2 case.

The Case L_1R_1S: In this case we begin with a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & \beta \\ \eta_2 & \gamma & \delta \end{array}$$

where the L_1 and R_1 definitions imply that $|\beta,\gamma|\neq 0$. Here $(x\eta_1+y\eta_2)^2=$ $=y[x(\beta+\gamma)+y\delta]$. Thus if $\{\beta+\gamma,\delta\}$ is independent, only scalar multiples of η_1 square to 0, while if $\{\beta+\gamma,\delta\}$ is dependent, there are two independent elements with this property. As above, we procced by considering cases: that $\{\beta+\gamma,\delta\}$ is independent, and that $\{\beta+\gamma,\delta\}$ is dependent.

(1) Assume first that $\{\beta+\gamma,\delta\}$ is independent.

(1a) $\{\eta_1,\beta\}$ is independent: Replacing η_2 by β, we arrive at a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_2 \\ \eta_2 & c_1\eta_1+c_2\eta_2 & d_1\eta_1+d_2\eta_2 \end{array}\quad \left(bc_1\neq 0,\ \begin{vmatrix} c_1 & b+c_2 \\ d_1 & d_2 \end{vmatrix}\neq 0\right).$$

Taking $\zeta_1=(1/b)\eta_1$, $\zeta_2=(1/c_1)\eta_2$, $c'=c_2b$, $d_1'=bd_1/c_1^2$ and $d_2'=d_2/c_1$, we arrive at the table:

$$(L_1R_1S/1)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_2 \\ \zeta_2 & \zeta_1+c'\zeta_2 & d_1'\zeta_1+d_2'\zeta_2 \end{array}\quad (d_2'-(1+c')d_1'\neq 0).$$

When $c'=-1$, $d_1'=0$, and $d_2'=1$, this is Wallace's algebra A_4.

1b) $\{\eta_1, \beta\}$ is dependent: Since $|\beta, \gamma| \neq 0$, we must have $\beta = b\eta_2$, $b \neq 0$, and hence $\{\eta_1, \gamma\}$ is independent. In this case replacing η_2 by γ yields a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b\eta_1 \\ \eta_2 & c\eta_2 & d_1\eta_1 + d_2\eta_2 \end{array} \quad \left(\begin{vmatrix} b & c \\ d_1 & d_2 \end{vmatrix} \neq 0 \right).$$

Here, taking $\zeta_1 = (1/c)\eta_1$, $\zeta_2 = (1/b)\eta_2$, $d_1' = cd_1/b^2$ and $d_2' = d_2/b$, we arrive at the table:

$$(L_1 R_1 S/2) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & \zeta_1 & d_1'\zeta_1 + d_2'\zeta_2 \end{array} \quad (d_1' \neq d_2').$$

(2) $\{\beta+\gamma, \delta\}$ is dependent: Since $|\beta, \gamma| \neq 0$ implies that $\beta+\gamma \neq 0$, $\delta = k(\beta+\gamma)$ for some $k \in \mathbf{R}$. Then $(-k\eta_1+\eta_2)^2 = -k(\beta+\gamma)+\delta = 0$, and replacing η_2 by $-k\eta_1 + {} + \eta_2$ yields a table in the form:

$$\begin{array}{c|cc} & \eta_1 & \eta_2 \\ \hline \eta_1 & 0 & b_1\eta_1 + b_2\eta_2 \\ \eta_2 & c_1\eta_1 + c_2\eta_2 & 0 \end{array} \quad \left(\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \neq 0 \right).$$

(2a) $b_2, c_1 \neq 0$, $b_1/c_1 < b_2/c_2$: Taking $\zeta_1 = (1/b_2)\eta_1$, $\zeta_2 = (1/c_1)\eta_2$, $b' = b_1/c_1$, and $c' = c_2/b_2$, we arrive at the table:

$$(L_1 R_1 S/3) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & b'\zeta_1 + \zeta_2 \\ \zeta_2 & \zeta_1 + c'\zeta_2 & 0 \end{array} \quad (b'c' < 1).$$

(2b) $b_2, c_1 \neq 0$, $b_1/c_1 > b_2/c_2$: Taking $\zeta_1 = (1/c_1)\eta_2$, $\zeta_2 = (1/b_2)\eta_1$, $b' = c_2/b_2$, and $c' = b_1/c_1$, we again arrive at a table of the form $(L_1 R_1 S/3)$.

(2c) $b_2 = 0$ $(b_1, c_2 \neq 0)$, $c_1 \neq 0$: Taking $\zeta_1 = (1/c_2)\eta_1$, $\zeta_2 = (1/b_1)\eta_2$, and $c' = c_1/b_1$ yields the table:

$$(L_1 R_1 S/4) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & c'\zeta_1 + \zeta_2 & 0 \end{array} \quad (c' \neq 0).$$

(2d) $c_1 = 0$ $(b_1, c_2 \neq 0)$, $b_2 \neq 0$: Taking $\zeta_1 = (1/b_1)\eta_2$, $\zeta_2 = (1/c_2)\eta_1$, and $c' = b_2/c_2$ also yields a table of the form $(L_1 R_1 S/4)$.

(2e) $b_2 = c_1 = 0$ $(b_1, c_2 \neq 0)$: We take $\zeta_1 = (1/c_2)\eta_1$ and $\zeta_2 = (1/b_1)\eta_2$ to get the table:

$$(L_1 R_1 S/5) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & 0 & \zeta_1 \\ \zeta_2 & \zeta_2 & 0 \end{array}.$$

The Case L_1R_1N: In this case we begin with a table in the form:

	η_1	η_2
η_1	α	0
η_2	γ	δ

where the L_1 and R_1 definitions imply that $|\alpha, \delta| \neq 0$ and hence $\{\alpha, \delta\}$ is independent. It follows that $\gamma = k\alpha + \ell\delta$, where $k = |\gamma, \delta|/|\alpha, \delta|$ and $\ell = |\alpha, \gamma|/|\alpha, \delta|$, and one sees that the N condition holds if and only if $k\ell \neq 1$, i.e., if and only if

$$|\gamma, \delta|\,|\alpha, \gamma| \neq |\alpha, \delta|^2.$$

Excepting scalar multiples, there are exactly two pairs of left and right zero divisors in the algebra: η_1, η_2 and $\bar{\eta}_1$, $\bar{\eta}_2$, where $\bar{\eta}_1 = -k\eta_1 + \eta_2$ and $\bar{\eta}_2 = \eta_1 - \ell\eta_2$; both are pairs of independent elements since $k\ell \neq 1$. With respect to $\{\eta_1, \eta_2\}$ the algebra has the multiplication table:

	η_1	η_2	
η_1	$a_1\eta_1 + a_2\eta_2$	0	
η_2	$c_1\eta_1 + c_2\eta_2$	$d_1\eta_1 + d_2\eta_2$	$(a_1d_2 - a_2d_1 \neq 0)$

and with respect to $\{\bar{\eta}_1, \bar{\eta}_2\}$, it has the table:

	$\bar{\eta}_1$	$\bar{\eta}_2$	
$\bar{\eta}_1$	$\bar{a}_1\bar{\eta}_1 + \bar{a}_2\bar{\eta}_2$	0	
$\bar{\eta}_2$	$\bar{c}_1\bar{\eta}_1 + \bar{c}_2\bar{\eta}_2$	$\bar{d}_1\bar{\eta}_1 + \bar{d}_2\bar{\eta}_2$	$(\bar{a}_1\bar{d}_2 - \bar{a}_2\bar{d}_1 \neq 0)$

where $\bar{a}_1 = \ell d_1 + d_2$, $\bar{a}_2 = d_1 + kd_2$, $\bar{c}_1 = -\ell c_1 - c_2$, $\bar{c}_2 = -c_1 - kc_2$, $\bar{d}_1 = \ell a_1 + a_2$, $\bar{d}_2 = a_1 + ka_2$, and hence $\begin{vmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{d}_1 & \bar{d}_2 \end{vmatrix} = (1 - k\ell) \begin{vmatrix} a_1 & a_2 \\ d_1 & d_2 \end{vmatrix} \neq 0$.

In this case we proceed by considering cases determined by whether the numbers c_1, c_2, $\bar{c}_1$, and $\bar{c}_2$ are zero or nonzero. While at first glance it would appear that there are sixteen such cases, the formulas $\bar{c}_1 = -\ell c_1 - c_2$ and $\bar{c}_2 = -c_1 - kd_2$ imply that only eight are actually possible. The remaining cases can be split into four groups as follows:

(1) $c_1 = c_2 = \bar{c}_1 = \bar{c}_2 = 0$.
(2) Exactly two of c_1, c_2, $\bar{c}_1$ and $\bar{c}_2$ are zero.
(These are necessarily either c_1 and $\bar{c}_2$, or c_2 and $\bar{c}_1$.)
(3) Exactly one of c_1, c_2, $\bar{c}_1$ and $\bar{c}_2$ is zero.
(4) All of c_1, c_2, $\bar{c}_1$ and $\bar{c}_2$ are nonzero.

(1) Assume first that $c_1 = c_2 = 0 = \bar{c}_1 = \bar{c}_2 = 0$.

(1a) $a_1=0$ $(a_2, d_1\neq 0)$: Taking $\zeta_1=a_2^{-2/3}d_1^{-1/3}\eta_1$, $\zeta_2=a_2^{-1/3}d_1^{-2/3}\eta_2$ and $d=a_2^{-1/3}d_1^{-2/3}d_2$, we arrive at a table in the form:

$$(L_1R_1N/1)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_2 & 0 \\ \zeta_2 & 0 & \zeta_1+d\zeta_2. \end{array}$$

(1b) $a_1\neq 0$, $d_2=0$ $(a_2, d_1\neq 0)$: Take $\zeta_1=a_2^{-1/3}d_1^{-2/3}\eta_2$, $\zeta_2=a_2^{-2/3}d_1^{-1/3}\eta_1$ and $d=a_1a_2^{-2/3}d_1^{-1/3}$ to again obtain a table in the form $(L_1R_1N/1)$.

(1c) a_1, $d_2\neq 0$: Taking $\zeta_1=(1/a_1)\eta_1$, $\zeta_2=(1/d_2)\eta_2$, $a=a_2d_2/a_1^2$ and $d=a_1d_1/d_2^2$, we arrive at the table:

$$(L_1R_1N/2)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1+a\zeta_2 & 0 \\ \zeta_2 & 0 & d\zeta_1+\zeta_2 \end{array} \quad (ad\neq 1).$$

(2) Next assume that exactly two of c_1, c_2, $\bar{c}_1$ and $\bar{c}_2$ are zero.

(2a) $c_1\neq 0$, $c_2=0$, $\bar{c}_1=0$ $(a_2=0,\ a_1\neq 0)$: Take $\zeta_1=(1/a_1)\eta_1$, $\zeta_2=(1/c_1)\eta_2$, $d_1'=a_1d_1/c_1^2$ and $d_2'=d_2/c_1$ to arrive at the table:

$$(L_1R_1N/3)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & \zeta_1 & 0 \\ \zeta_2 & \zeta_1 & d_1'\zeta_1+d_2'\zeta_2 \end{array} \quad (d_2'\neq 0).$$

(2b) $c_1=0$, $c_2\neq 0$, $\bar{c}_2=0$ $(d_1=0,\ d_2\neq 0)$: Take $\zeta=(1/d_2)\eta_2$, $\zeta_2=(-1/c_2)\eta_1+(1/d_2)\eta_2$, $d_1'=(a_1c_2-a_1d_2)/c_2^2$ and $d_2'=-a_1/c_2$ to again get a table in the form $(L_1R_1N/3)$.

(3) Next assume that exactly one of c_1, c_2, $\bar{c}_1$ and $\bar{c}_2$ is zero.

(3a) $c_1=0$, $c_2\neq 0$, $\bar{c}_1\neq 0$, $\bar{c}_2\neq 0$ $(d_1\neq 0)$: Take

$$\begin{aligned}
\zeta_1 &= (1/c_2)\eta_1+[(a_1d_2-a_2d_1)/c_2^2d_1]\eta_2,\\
\zeta_2 &= (-1/c_2)\eta_1+[a_1/(a_1d_2-a_2d_1)]\eta_2,\\
a_1' &= (a_1c_2d_1+a_1d_2^2-a_2d_1d_2)/c_2^2d_1,\\
a_2' &= (a_1d_2-a_2d_1)(-a_1d_1d_2+a_2d_1^2+c_2d_1d_2)/c_2^3d_1^2,\\
d_1' &= (a_1^2c_2d_1+a_1a_2d_1d_2-a_2^2d_1^2)/(a_1d_2-a_2d_1)^2 \quad \text{and}\\
d_2' &= (-a_1^2d_2+a_1a_2d_1+a_2c_2d_1)/c_2(a_1d_2-a_2d_1)
\end{aligned}$$

to arrive at the table:

$$(^*)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a_1'\zeta_1+a_2'\zeta_2 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d_1'\zeta_1+d_2'\zeta_2 \end{array} \quad [a_1'(d_2'+1)-a_2'(d_1'+1)=0,\ a_1'-a_2'\neq d_2'-d_1']$$

which is of the form:

$$(L_1R_1N/4)\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a_1\zeta_1+a_2\zeta_2 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d_1\zeta_1+d_2\zeta_2 \end{array} \quad (|\alpha, \delta| \neq 0,\ (k+1)(\ell+1)=0,\ k\ell \neq 1)$$

where, as defined above,

$$k = (d_2-d_1)/(a_1d_2-a_2d_1) \quad \text{and} \quad \ell = (a_1-a_2)/(a_1d_2-a_2d_1).$$

(3b) $c_1 \neq 0$, $c_2=0$, $\bar{c}_1 \neq 0$, $\bar{c}_2 \neq 0$ $(a_2 \neq 0)$: Taking

$$\zeta_1 = [d_2/(a_1d_2-a_2d_1)]\eta_1-(1/c_1)\eta_2,$$
$$\zeta_2 = [(a_1d_2-a_2d_1)/a_2c_1^2]\eta_1+(1/c_1)\eta_2,$$
$$a_1' = (-a_1d_2^2+a_2c_1d_1+a_2d_1d_2)/c_1(a_1d_2-a_2d_1),$$
$$a_2' = (a_1a_2d_1d_2-a_2^2d_1^2+a_2c_1d_2^2)/(a_1d_2-a_2d_1)^2,$$
$$d_1' = (a_1d_2-a_2d_1)(a_1c_1-a_1d_2+a_2d_1)/a_2c_1^3, \quad \text{and}$$
$$d_2' = (a_1^2d_2-a_1a_2d_1+a_2c_1d_2)/a_2c_1^2,$$

we arrive at the table:

$$(^{**})\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a_1'\zeta_1+a_2'\zeta_2 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d_1'\zeta_1+d_2'\zeta_2 \end{array} \quad [(a_1'+1)d_2'-(a_2'+1)d_1'=0,\ a_1'-a_2' \neq d_2'-d_1']$$

which is again in the form $(L_1R_1N/4)$.

(3c) $c_1 \neq 0$, $c_2 \neq 0$, $\bar{c}_1=0$, $\bar{c}_2 \neq 0$ $(a_1c_1c_2-a_2c_1^2+a_1c_2d_2-a_2c_2d_1=0)$:

(3ci) $a_1c_1+a_1d_2-a_2d_1 \neq 0$ $(c_2=a_2c_1^2/(a_1c_1+a_1d_2-a_2d_1),\ a_2 \neq 0)$:

Take

$$\zeta_1 = [(a_1c_1+a_1d_2-a_2d_1)/a_2c_1^2]\eta_1, \quad \zeta_2 = (1/c_1)\eta_2,$$
$$a_1' = a_1(a_1c_1+a_1d_2-a_2d_1)/a_1c_1^2, \quad a_2' = (a_1c_1+a_1d_2-a_2d_1)^2/a_1c_1^3,$$
$$d_1' = d_1a_2/(a_1c_1+a_1d_2-a_2d_1), \quad \text{and} \quad d_2' = d_2/c_1$$

to arrive at a table in the form (*) and hence of the form $(L_1R_1N/4)$.

(3cii) $a_1c_1+a_1d_2-a_2d_1=0$ $(a_2=0,\ a_1 \neq 0,\ d_2 \neq 0,\ c_1=-d_2)$: Take $\zeta_1=(1/c_2)\eta_1$, $\zeta_2=(-1/d_2)\eta_2$, $a'=a_1/c_2$, $d_1'=c_2d_1/d_2^2$ to arrive at a table in the form:

$$(^{***})\quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a'\zeta_1 & 0 \\ \zeta_2 & \zeta_1+\zeta_2 & d'\zeta_1-\zeta_2 \end{array} \quad (a'd' \neq 0,\ a'+d' \neq -1)$$

which is also of the form $(L_1R_1N/4)$.

(3d) $c_1 \neq 0$, $c_2 \neq 0$, $\bar{c}_1 \neq 0$, $\bar{c}_2 = 0$ $(a_1 c_1 d_2 - a_2 c_1 d_1 + c_1 c_2 d_2 - c_2^2 d_1 = 0)$:
(3di) $a_1 d_2 - a_2 d_1 + c_2 d_2 \neq 0$ $(c_1 = c_2^2 d_1/(a_1 d_2 - a_2 d_1 + c_2 d_2),\ d_1 \neq 0)$: Take

$$\zeta_1 = (1/c_2)\eta_1, \quad \zeta_2 = [(a_1 d_2 - a_2 d_1 + c_2 d_2)/c_2^2 d_1]\eta_2,$$

$$a_1' = a_1/c_2, \quad a_2' = a_2 d_1/(a_1 d_2 - a_2 d_1 + c_2 d_2),$$

$$d_1' = (a_1 d_2 - a_2 d_1 + c_2 d_2)^2/c_2^3 d_1, \quad \text{and} \quad d_2' = d_2(a_1 d_2 - a_2 d_1 + c_2 d_2)/c_2^2 d_1$$

to arrive at a table in the form (**) and hence of the form $(L_1 R_1 N/4)$.

(3dii) $a_1 d_2 - a_2 d_1 + c_2 d_2 = 0$ $(d_1 = 0,\ a_1 \neq 0,\ d_2 \neq 0,\ c_2 = -a_1)$: Take $\zeta_1 = (-1/a_1)\eta_1$, $\zeta_2 = (1/c_1)\eta_2$, $a' = a_2 c_1/a_1^2$, and $d' = d_2/c_1$ to arrive at the table:

$$(****) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & -\zeta_1 + a'\zeta_2 & 0 \\ \zeta_2 & \zeta_1 + \zeta_2 & d'\zeta_2 \end{array} \quad (a' \neq 0,\ d' \neq 0,\ d \neq -1 - a')$$

which is also of the form $(L_1 R_1 N/4)$.

(4) Finally, assume that $c_1, c_2, \bar{c}_1, \bar{c}_2 \neq 0$. Then $a_1 c_1 c_2 - a_2 c_1^2 + a_1 c_2 d_2 - a_2 c_2 d_1 \neq 0$, $a_1 c_1 d_2 - a_2 c_1 d_1 + c_1 c_2 d_2 - c_2^2 d_1 \neq 0$. Taking $\zeta_1 = (1/c_2)\eta_1$, $\zeta_2 = (1/c_1)\eta_2$, $a_1' = a_1/c_2$, $a_2' = a_2 c_1/c_1^2$, $d_1' = c_2 d_1/c_1^2$, and $d_2' = d_2/c_1$, we arrive at the table:

$$(L_1 R_1 N/5) \quad \begin{array}{c|cc} & \zeta_1 & \zeta_2 \\ \hline \zeta_1 & a_1'\zeta_1 + a_2'\zeta_2 & 0 \\ \zeta_2 & \zeta_1 + \zeta_2 & d_1'\zeta_1 + d_2'\zeta_2 \end{array} \quad \begin{array}{l} [a_1'(d_1'+1) - a_2'(d_1'+1) \neq 0, \\ (a_1'+1)d_2' - (a_2'+1)d_1' \neq 0, \\ a_1' d_2' - a_2' d_1' \neq 0]. \end{array}$$

On the other hand, because $k\ell \neq 1$, we may also use the basis change:

$$\zeta_1 = [(c_1 d_2 - c_2 d_1)/(a_1 c_1 d_2 - a_2 c_1 d_1 + c_1 c_2 d_2 - c_2^2 d_1)]\eta_1 -$$
$$- [(a_1 d_2 - a_2 d_1)/(a_1 c_1 d_2 - a_2 c_1 d_1 + c_1 c_2 d_2 - c_2^2 d_1)]\eta_2 \quad \text{and}$$
$$\zeta_2 = - [(a_1 d_2 - a_2 d_1)/(a_1 c_1 c_2 - a_2 c_1^2 + a_1 c_2 d_2 - a_2 c_2 d_1)]\eta_1 -$$
$$+ (a_1 c_2 - a_2 c_1)/(a_1 c_1 c_2 - a_2 c_1^2 + a_1 c_2 d_2 - a_2 c_2 d_1)]\eta_2$$

to get another table of the form $(L_1 R_1 N/5)$, where in this case,

$$a_1' = -(a_1 c_2 d_1 - a_2 c_1 d_1 + a_1 d_2^2 - a_2 d_1 d_2)/(a_1 c_1 d_2 - a_2 c_1 d_1 + c_1 c_2 d_2 - c_2^2 d_1),$$
$$a_2' = -(a_1 c_1 c_2 - a_2 c_1^2 + a_1 c_2 d_2 - a_2 c_2 d_1)(a_1 d_1 d_2 - a_2 d_1^2 + c_1 d_2^2 - c_2 d_1 d_2) \cdot$$
$$\cdot (a_1 c_1 d_2 - a_2 c_1 d_1 + c_1 c_2 d_2 - c_2^2 d_1)^{-2},$$
$$d_1' = -(a_1^2 c_2 - a_1 a_2 c_1 + a_1 a_2 d_2 - a_2^2 d_1)(a_1 c_1 d_2 - a_2 c_1 d_1 + c_1 c_2 d_3 - c_2^2 d_1) \cdot$$
$$\cdot (a_1 c_1 c_2 - a_2 c_1^2 + a_1 c_2 d_2 - a_2 c_2 d_1)^{-2},$$

$$\text{and} \quad d_2' = -(a_1^2 d_2 - a_1 a_2 d_1 + a_2 c_1 d_2 - a_2 c_2 d_1)/(a_1 c_1 c_2 - a_2 c_1^2 + a_1 c_2 d_2 - a_2 c_2 d_1).$$

In this case, two tables:

	ζ_1	ζ_2
ζ_1	$a_1\zeta_1+a_2\zeta_2$	0
ζ_2	$\zeta_1+\zeta_2$	$d_1\zeta_1+d_2\zeta_2$

	ζ_1'	ζ_2'
ζ_1'	$a_1'\zeta_1+a_2'\zeta_2$	0
ζ_2'	$\zeta_1'+\zeta_2'$	$d_1'\zeta_1'+d_2'\zeta_2'$

satisfying the side conditions defining the $(L_1R_1N/5)$ case are tables for isomorphic algebras if and only if

$$(^\#)\quad a_1' = a_1,\quad a_2' = a_2,\quad d_1' = d_1,\quad d_2' = d_2,\quad \text{or}$$

$$(^{\#\#})\quad a_1' = -(a_1d_1-a_2d_1+a_1d_2^2-a_2d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1),$$

$$a_2' = -(a_1-a_2+a_1d_2-a_2d_1)(a_1d_1d_2-a_2d_1^2+d_2^2-d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1)^2,$$

$$d_1' = -(a_1^2-a_1a_2+a_1a_2d_2-a_2^2d_1)(a_1d_2-a_2d_1+d_2-d_1)/(a_1-a_2+a_1d_2-a_2d_1)^2,$$

$$\text{and}\quad d_2' = -(a_1^2d_2-a_1a_2d_1+a_2d_2-a_2d_1)/(a_1-a_2+a_1d_2-a_2d_1).$$

The case $(L_1R_1N/5)$ need not necessarily result in two distinct tables since the systems $(^\#)$ and $(^{\#\#})$ may yield the same solutions, as is true, for example, in the case that $a_1=2$, $a_2=-10$, $d_1=3$, and $d_2=-3$. Clearly $(^\#)$ and $(^{\#\#})$ have the same solutions if and only if

$$a_1 = -(a_1d_1-a_2d_1+a_1d_2^2-a_2d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1),$$

$$a_2 = -(a_1-a_2+a_1d_2-a_2d_1)(a_1d_1d_2-a_2d_1^2+d_2^2-d_1d_2)/(a_1d_2-a_2d_1+d_2-d_1)^2,$$

$$d_1 = -(a_1^2-a_1a_2+a_1a_2d_2-a_2^2d_1)(a_1d_2-a_2d_1+d_2-d_1)/(a_1-a_2+a_1d_2-a_2d_1)^2,$$

$$\text{and}\quad d_2 = -(a_1^2d_2-a_1a_2d_1+a_2d_2-a_2d_1)/(a_1-a_2+a_1d_2-a_2d_1).$$

Each of the first and fourth of these equations is equivalent to

$$a_1+d_2+1 = 0,$$

and assuming this equation holds, the second and third equation each can be shown to be equivalent to

$$(a_2+d_1+1)\,|\alpha,\delta|+2(a_1-a_2)(d_1-d_2) = 0.$$

If these two equations are not both true, an algebra in the case $(L_1R_1N/5)$ will have two distinct tables fitting the canonical form; if they do hold, there is only one such table.

We have shown that there are ten distinct canonical forms for tables in the L_1R_1 case.

Summary. This paper proves that there are 34 distinct table forms two-dimensionel real algebras. Four of these describe division algebras and are classified in [1]. The other 30 describe algebras with zero divisors and appear above.

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